The monomial basis and the Q-basis of the Hopf algebra of parking functions

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Abstract

Consider the vector space $\mathbb{K}\mathcal{P}$ spanned by parking functions. By representing parking functions as labeled digraphs, Hivert, Novelli and Thibon constructed a cocommutative Hopf algebra PQSym* on $\mathbb{K}\mathcal{P}$. The product and coproduct of PQSym* are analogous to the product and coproduct of the Hopf algebra NCSym of symmetric functions in noncommuting variables defined in terms of the power sum basis. In this paper, we view a parking function as a word. We shall construct a Hopf algebra PFSym on $\mathbb{K}\mathcal{P}$ with a formal basis $\{M_a\}$ analogous to the monomial basis of NCSym. By introducing a partial order on parking functions, we transform the basis $\{M_a\}$ to another basis $\{Q_a\}$ via the Möbius inversion. We prove the freeness of PFSym by finding two free generating sets in terms of the M-basis and the Q-basis, and we show that PFSym is isomorphic to the Hopf algebra PQSym*. It turns out that our construction, when restricted to permutations and non-increasing parking functions, leads to a new way to approach the Grossman-Larson Hopf algebras of ordered trees and heap-ordered trees.

Keywords: Hopf algebra; parking function; partition; symmetric functions in noncommuting variables

1 Introduction

The algebra NCSym of symmetric functions in noncommuting variables was first studied by Wolf [18] in 1936. Rosas and Sagan [16] have found bases of NCSym analogous to the monomial, elementary, homogeneous and power sum bases of the algebra of symmetric functions in commuting variables. All these bases of NCSym are indexed by set partitions. The Hopf algebra structure on NCSym was introduced by Bergeron, Reutenauer, Rosas and Zabrocki [5].

Consider the vector space $\mathbb{K}\mathcal{P}$ spanned by parking functions. By representing parking functions as labeled digraphs, Hivert, Novelli and Thibon [10] introduced a Hopf algebra PQSym on $\mathbb{K}\mathcal{P}$ in their study of commutative and cocommutative Hopf algebras based on various combinatorial structures. The graded dual PQSym* of PQSym contains a Hopf subalgebra isomorphic to NCSym. This fact can be shown easily from expressions for the product and the coproduct of NCSym in terms of the power sum basis [4]. To be more specific, the product of PQSym* is given by shifted concatenation of parking functions and the coproduct encodes all ways to divide connected components of a parking function into two parts and relabel each part. Similarly, in terms of the power sum basis, the product of NCSym is given by shifted union of set partitions and the coproduct is given by dividing all blocks of a partition into two

parts and then standardize each part. In this paper, we are concerned with the problem of constructing a Hopf algebra on $\mathbb{K}\mathcal{P}$ with a basis analogous to the monomial basis of NCSym. Using a formal basis $\{M_a\}$ indexed by parking functions, we define a product $\star$ and a coproduct Δ on $\mathbb{K}\mathcal{P}$ and we show that $\mathrm{PFSym} = (\mathbb{K}\mathcal{P}, \star, \Delta)$ is a Hopf algebra isomorphic to PQSym^*. As will be seen, the basis $\{M_a\}$ of PFSym is a natural generalization of the monomial basis of NCSym.

To define the product $\star$ and the coproduct Δ of PFSym, we view parking functions as words on positive integers. In this notation, a parking function can be decomposed into subwords via its left-to-right minima. Then the product $\star$ can be defined in terms of matchings between such decompositions and the coproduct Δ can be defined by dividing the decomposition into two parts. Also, based on this decomposition, we define the slash product and the split product on parking functions, and then further introduce the concepts of atomic parking functions and unsplitable parking functions. We shall show that the Hopf algebra PFSym is free by finding two free generating sets. The first one consists of those basis elements M_α indexed by unsplitable parking functions. The second one, indexed by atomic parking functions, consists of elements from another basis $\{Q_a\}$. The Q-basis for PFSym is related to the basis $\{M_a\}$ via the Möbius inversion on a partial order on parking functions. This basis is a natural analog of the q-basis of the algebra NCSym introduced by Bergeron and Zabrocki [6].

This paper is organized as follows. In Section 2, we give an overview of the Hopf algebra NCSym of symmetric functions in noncommuting variables. The product of NCSym will be described in terms of matchings between set partitions. In Section 3, we introduce the LR-decomposition of a parking function. By means of this decomposition and a formal basis $\{M_a\}$ indexed by parking functions, we define a product $\star$ and a coproduct Δ on $\mathbb{K}\mathcal{P}$ and we show that $\mathrm{PFSym} = (\mathbb{K}\mathcal{P}, \star, \Delta)$ is a Hopf algebra. In Section 4, we introduce another basis $\{Q_a\}$ via a partial order on parking functions and show the freeness of PFSym in terms of the M-basis and the Q-basis. In the last section, we consider Hopf subalgebras of PFSym. In addition to a Hopf subalgebra isomorphic to NCSym, we find Hopf subalgebras isomorphic to the Grossman-Larson Hopf algebras of ordered trees and heap-ordered trees [8].

2 The Hopf algebra NCSym

In this section, we give an overview of the Hopf algebra NCSym of symmetric functions in noncommuting variables. This Hopf algebra was introduced by Bergeron, Reutenauer, Rosas and Zabrocki [5] in their study of the connections between the algebra of symmetric functions in commuting variables and the algebra of symmetric functions in noncommuting variables. Instead of employing the lattice structure of partitions, we use matchings between partitions to describe the product of NCSym with respect to the monomial basis. As will be seen, this description is of importance in providing us ideas to construct the Hopf algebra PFSym of parking functions.

Throughout this paper, $\mathbb{K}$ stands for a field of characteristic zero and $\mathbb{K}S$ denotes the vector space with the set S as basis. For the background on Hopf algebras, see Abe [1] and Sweedler [17].

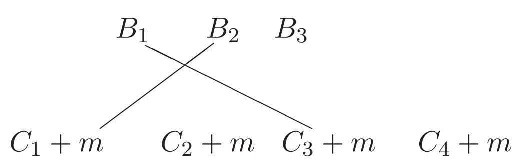
The Hopf algebra NCSym can be formally described in terms of partitions. Let $[n] =$

$\{1, 2, \ldots, n\}$. A partition of $[n]$ is a set $\{B_1, B_2, \ldots, B_k\}$ of pairwise disjoint nonempty subsets of $[n]$ whose union is $[n]$. The subsets B_i are called blocks of π. Without loss of generality, we may assume that the blocks of a partition are arranged in the increasing order of their minimal elements and the elements in each block are written in increasing order. Denote by Π_n the set of partitions of $[n]$. Set $\Pi_0 = \{\emptyset\}$ and $\Pi = \cup_{n\geq 0}\Pi_n$. Let

$$\mathrm{NCSym} = \bigoplus_{n\geq 0} \mathbb{K}\{M_\pi : \pi \in \Pi_n\},$$

where $\{M_\pi\}$ is a formal basis called monomial basis and the space $\mathbb{K}\{M_\emptyset\}$ is regarded as $\mathbb{K}$ by identifying $M_\emptyset$ with 1.

The product of NCSym is defined as follows. Firstly, for any $\pi \in \Pi$, we set $M_\pi M_\emptyset = M_\emptyset M_\pi = M_\pi$. Suppose that $m, n \geq 1$, $\pi = \{B_1, B_2, \ldots, B_r\} \in \Pi_m$ and $\sigma = \{C_1, C_2, \ldots, C_s\} \in \Pi_n$. Let $\sigma + m = \{C_1 + m, C_2 + m, \ldots, C_s + m\}$, where $C_i + m$ denotes the set obtained by adding m to each element in C_i. We shall consider the set $R(\pi, \sigma)$ of all matchings between π and $\sigma+m$, where the blocks of π and $\sigma+m$ are considered as vertices. Here a matching between two vertex sets X and Y means a bipartite graph with bipartition (X, Y) such that there are no two edges having a common vertex. For example, if $\pi = \{B_1, B_2, B_3\}$ and $\sigma = \{C_1, C_2, C_3, C_4\}$, the following diagram represents a matching with two edges $(B_1, C_3 + m)$ and $(B_2, C_1 + m)$.



Given a matching in $R(\pi, \sigma)$, we can construct a partition from the blocks of π and $\sigma + m$ by combining blocks B_i and $C_j + m$ if they form an edge of the matching. Let $S(\pi, \sigma)$ denote the set of such partitions obtained from matchings in $R(\pi, \sigma)$. Then the product of NCSym is defined by

$$M_\pi M_\sigma = \sum_{\tau \in S(\pi,\sigma)} M_\tau. \tag{2.1}$$

For example, we have

$$\begin{aligned} M_{\{\{1,3\},\{2,4\}\}} M_{\{\{1,3,4\},\{2\}\}} =& M_{\{\{1,3\},\{2,4\},\{5,7,8\},\{6\}\}} + M_{\{\{1,3,5,7,8\},\{2,4\},\{6\}\}} \\ &+ M_{\{\{1,3\},\{2,4,5,7,8\},\{6\}\}} + M_{\{\{1,3,6\},\{2,4\},\{5,7,8\}\}} \\ &+ M_{\{\{1,3\},\{2,4,6\},\{5,7,8\}\}} + M_{\{\{1,3,5,7,8\},\{2,4,6\}\}} \\ &+ M_{\{\{1,3,6\},\{2,4,5,7,8\}\}}. \end{aligned}$$

To define the coproduct on NCSym, we need the notion of standardization. Suppose that $\tau = \{B_1, B_2, \ldots, B_k\}$ is a family of disjoint nonempty finite sets of integers. Define the standardization of τ, denoted by $\mathrm{st}(\tau)$, to be the set partition obtained from τ by substituting the smallest element by 1, the second smallest element by 2, and so on. By convention, we let $\mathrm{st}(\emptyset) = \emptyset$. For example, let $\tau = \{\{2, 5, 7\}, \{4, 8\}, \{6, 9\}\}$. Then we have $\mathrm{st}(\tau) = \{\{1, 3, 5\}, \{2, 6\}, \{4, 7\}\}$.

The coproduct

$$\Delta\colon \mathrm{NCSym} \longrightarrow \mathrm{NCSym} \otimes \mathrm{NCSym}$$

is defined by

$$\Delta(M_\pi) = \sum_{\pi_1 \uplus \pi_2 = \pi} M_{\mathrm{st}(\pi_1)} \otimes M_{\mathrm{st}(\pi_2)}, \tag{2.2}$$

where by the notation

$$\sum_{\pi_1 \uplus \pi_2 = \pi}$$

we mean that the sum ranges over ordered pairs (π_1, π_2) of disjoint subsets of π such that $\pi_1 \cup \pi_2 = \pi$, see [11].

For example, we have

$$\begin{aligned}\Delta(M_{\{\{1,4,6\},\{2,5\},\{3\}\}}) =& 1 \otimes M_{\{\{1,4,6\},\{2,5\},\{3\}\}} + M_{\{\{1,2,3\}\}} \otimes M_{\{\{1,3\},\{2\}\}} \\ &+ M_{\{\{1,2\}\}} \otimes M_{\{\{1,3,4\},\{2\}\}} + M_{\{\{1\}\}} \otimes M_{\{\{1,3,5\},\{2,4\}\}} \\ &+ M_{\{\{1,3\},\{2\}\}} \otimes M_{\{\{1,2,3\}\}} + M_{\{\{1,3,4\},\{2\}\}} \otimes M_{\{\{1,2\}\}} \\ &+ M_{\{\{1,3,5\},\{2,4\}\}} \otimes M_{\{\{1\}\}} + M_{\{\{1,4,6\},\{2,5\},\{3\}\}} \otimes 1.\end{aligned}$$

Equipped with the product and coproduct defined as above, NCSym is a graded cocommutative Hopf algebra with unit $M_\emptyset = 1 \in \mathbb{K}$ and counit given by

$$\varepsilon(M_\pi) = \begin{cases} 1, & \text{if } \pi = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

3 The Hopf algebra PFSym of parking functions

In this section, based on a formal basis analogous to the monomial basis of NCSym, we define a product $\star$ and a coproduct Δ on the vector space $\mathbb{K}\mathcal{P}$ spanned by parking functions. We show that, with these operations, $\mathbb{K}\mathcal{P}$ is indeed a Hopf algebra.

We begin by introducing some basic notation and terminology. Let $[n]^k$ denote the set of words of length k on $[n]$. A word $a = a_1a_2\cdots a_n \in [n]^n$ is called a parking function if its nondecreasing rearrangement $b_1b_2\cdots b_n$ satisfies $b_i \leq i$ for $1 \leq i \leq n$. Let ϵ denote the unique empty word of length 0. Denote by P_n the set of parking functions of length n and set $P = \cup_{n\geq 0} P_n$, where $P_0 = \{\epsilon\}$.

Suppose that $a = a_1a_2\cdots a_n$ is a word on positive integers and $1 \leq i \leq n$. We say that i is a position of left-to-right minimum of a if $i = 1$ or $i \geq 2$ and $a_i < a_j$ for any $j < i$. Let $lr(a) = \{i_1, i_2, \ldots, i_s\}$ denote the set of positions of left-to-right minima of a, where $i_1 < i_2 < \cdots < i_s$. Set $i_{s+1} = n+1$. For $1 \leq j \leq s$, let

$$w_j = a_{i_{s+1-j}} a_{i_{s+1-j}+1} \cdots a_{i_{s+2-j}-1}.$$

We call the sequence $(w_1, w_2, \ldots, w_s)$ of words the LR-decomposition of a, denoted by F_a.

For example, let $a = 56357622315 \in P_{11}$. Then we have $lr(a) = \{1, 3, 7, 10\}$ and

$$F_a = (15, 223, 3576, 56).$$

Remark 3.1 *For a word a, let $\min(a)$ denote the minimal integer appearing in a. A word $a = a_1a_2 \ldots a_k$ is said to be dominated if $a_1 = \min(a)$. Let*

$$\mathcal{F}_n = \{F_a \mid a \in P_n\},\ n \geq 1$$

and set $\mathcal{F}_0 = \{(\,)\} = \{F_\epsilon\}$. It is easy to verify that a sequence $F = (w_1, w_2, \ldots, w_k) \in \mathcal{F}_n$ if and only if

(1) the word $w_k \cdots w_2 \cdot w_1$ obtained by concatenating w_i is a parking function;

(2) for $1 \leq i \leq k$, the word w_i is dominated;

(3) $\min(w_1) < \min(w_2) < \cdots < \min(w_k)$.

Moreover, for each $n \geq 0$, $a \mapsto F_a$ is a bijection between P_n and $\mathcal{F}_n$.

In the following, unless otherwise specified, we always identify a sequence $F = (w_1, w_2, \ldots, w_k)$ of dominated words such that $\min(w_1) < \min(w_2) < \cdots < \min(w_k)$ with the underlying set $\{w_1, w_2, \ldots, w_k\}$, since we can assume that the words in the set are arranged in increasing order of their minimal elements. So the notations such as $w_i \in F$, $F \cap G$, $F \cup G$, and $F \setminus G$ make sense.

Let

$$\mathbb{K}\mathcal{P} = \bigoplus_{n \geq 0} \mathbb{K}\{M_a : a \in P_n\},$$

where M_a denotes a formal basis element indexed by a parking function a. We call the basis $\{M_a \mid a \in P\}$ monomial basis of $\mathbb{K}\mathcal{P}$ and regard $\mathbb{K}\{M_\epsilon\}$ as $\mathbb{K}$ by identifying M_ϵ with 1.

The product on $\mathbb{K}\mathcal{P}$ is a natural generalization of that in NCSym, as described in Section 2. Firstly, set $M_a \star M_\epsilon = M_\epsilon \star M_a = M_a$ for any parking function a. Now suppose that $m \geq 1, n \geq 1$, $a = a_1a_2 \cdots a_m \in P_m$ and $b = b_1b_2 \cdots b_n \in P_n$. Let

$$F_a = (u_1, u_2, \ldots, u_r), \quad F_b = (w_1, w_2, \ldots, w_s)$$

and

$$F_b + m = (w_1 + m, w_2 + m, \ldots, w_s + m),$$

where $w_i + m$ denotes the word obtained by adding m to each integer in w_i. Let $R(a, b)$ denote the set of matchings between F_a and $F_b + m$. Given a matching Θ in $R(a, b)$, we define a set $F_a\Theta F_b$ of words from F_a and $F_b + m$ by concatenating words u_i and $w_j + m$ if they form an edge in Θ. Obviously, we have $F_a\Theta F_b \in \mathcal{F}_{m+n}$ and so we get a unique parking function, denoted by $a\Theta b$, whose $LR-$decomposition is $F_a\Theta F_b$. Define the product of M_a and M_b by

$$M_a \star M_b = \sum_{\Theta \in R(a,b)} M_{a\Theta b}. \tag{3.3}$$

For example, let

$$a = 211,\ \ b = 353112.$$

Then we have

$$F_a = (11, 2),\ \ F_b + 3 = (445, 686)$$

and

$$\begin{aligned}\{F_a \Theta F_b \colon \Theta \in R(a,b)\} = \{&(11, 2, 445, 686), (11, 2686, 445), (11686, 2, 445),\\ &(11, 2445, 686), (11445, 2, 686), (11445, 2686), (11686, 2445)\}.\end{aligned}$$

So we have

$$\begin{aligned}M_a \star M_b = &M_{686445211} + M_{445268611} + M_{445211686} + M_{686244511}\\ &+ M_{686211445} + M_{268611445} + M_{244511686}.\end{aligned}$$

Now we proceed to the definition of the coproduct map

$$\Delta \colon \mathbb{K}\mathcal{P} \longrightarrow \mathbb{K}\mathcal{P} \otimes \mathbb{K}\mathcal{P}.$$

To this end, we shall use the parkization $\mathrm{Park}(a)$ of a word a, which was used by J.-C. Novelli and J.-Y. Thibon in [14]. For completeness, let us give an overview of the definition of $\mathrm{Park}(a)$. Suppose that $a = a_1 a_2 \cdots a_n$ is a word on positive integers. Set

$$d(a) = \min\{i :\ \ |\{j : a_j \le i\}| < i\}.$$

If $d(a) = n + 1$, then we find that a is a parking function and let $\mathrm{Park}(a) = a$. Otherwise, let $\bar{a}$ be the word obtained from a by decrementing all the elements greater than $d(a)$ and let $\mathrm{Park}(a) = \mathrm{Park}(\bar{a})$. Since $\bar{a}$ is smaller than a in the lexicographic order, the algorithm terminates and always returns a parking function $\mathrm{Park}(a)$, called the parkization of a. It can be seen that when a is a word without repeated letters, $\mathrm{Park}(a)$ coincides with the standardization of a, which is obtained from a by substituting the smallest element by 1, the second smallest element by 2, and so on. Set $\mathrm{Park}(\epsilon) = \epsilon$.

For example, we give the algorithm of computing $\mathrm{Park}(875221)$ as follows:

$$\begin{aligned} a &= 875221 \qquad & d(a) &= 4\\ \bar{a} &= 764221 \qquad & d(\bar{a}) &= 5\\ \bar{\bar{a}} &= 654221 \qquad & d(\bar{\bar{a}}) &= 7,\end{aligned}$$

so we have $\mathrm{Park}(875221) = \mathrm{Park}(764221) = \mathrm{Park}(654221) = 654221$.

Now the coproduct Δ is defined by

$$\Delta(M_a) = \sum_{F_{a'} \cup F_{a''} = F_a} M_{\mathrm{Park}(a')} \otimes M_{\mathrm{Park}(a'')}, \tag{3.4}$$

where the sum ranges over ordered pairs (a', a'') of subwords of a such that $F_{a'} \cap F_{a''} = \emptyset$ and $F_{a'} \cup F_{a''} = F_a$.

For example, let $a = 445132$, then we have $F_a = (132, 445)$ and

$$\begin{aligned}\Delta(M_a) =& M_\epsilon \otimes M_{445132} + M_{\mathrm{Park}(445)} \otimes M_{\mathrm{Park}(132)}\\ &+ M_{\mathrm{Park}(132)} \otimes M_{\mathrm{Park}(445)} + M_{445132} \otimes M_\epsilon\\ =& 1 \otimes M_{445132} + M_{112} \otimes M_{132} + M_{132} \otimes M_{112} + M_{445132} \otimes 1.\end{aligned}$$

Let $\mathrm{PFSym} = (\mathbb{K}\mathcal{P}, \star, \Delta)$ denote the vector space $\mathbb{K}\mathcal{P}$ equipped with the product $\star$ and the coproduct Δ.

Theorem 3.2 PFSym *is a cocommutative Hopf algebra with unit M_ϵ and counit given by*

$$\varepsilon(M_a) = \begin{cases} 1, & \textit{if } a = \epsilon; \\ 0, & \textit{otherwise.} \end{cases} \tag{3.5}$$

We will prove Theorem 3.2 through the following steps.

Proposition 3.3 *With the product $\star$ defined by* (3.3)*, $\mathbb{K}\mathcal{P}$ is an associative algebra.*

Proof. Suppose that $a \in P_l$, $b \in P_m$ and $c \in P_n$. Let $F_a = (u_1, u_2, \ldots, u_r)$, $F_b = (v_1, v_2, \ldots, v_s)$ and $F_c = (w_1, w_2, \ldots, w_t)$. We need to show that

$$(M_a \star M_b) \star M_c = M_a \star (M_b \star M_c). \tag{3.6}$$

By definition, we have

$$(M_a \star M_b) \star M_c = \sum_{\Theta \in R(a,b)} \sum_{\Lambda \in R(a\Theta b, c)} M_{(a\Theta b)\Lambda c}.$$

and

$$M_a \star (M_b \star M_c) = \sum_{\Theta' \in R(b,c)} \sum_{\Lambda' \in R(a, b\Theta' c)} M_{a\Lambda'(b\Theta' c)}.$$

To prove (3.6), it suffices to find a bijection

$$\begin{gathered}\{(\Theta, \Lambda) \mid \Theta \in R(a,b), \Lambda \in R(a\Theta b, c)\} \longrightarrow \{(\Theta', \Lambda') \mid \Theta' \in R(b,c), \Lambda' \in R(a, b\Theta' c)\}\\ (\Theta, \Lambda) \longmapsto (\Theta', \Lambda')\end{gathered}$$

such that $(a\Theta b)\Lambda c = a\Lambda'(b\Theta' c)$. Suppose that $\Theta \in R(a,b)$ and $\Lambda \in R(a\Theta b, c)$. Let Θ' be the matching between F_b and $F_c + m$ whose edges are those $(v_j, w_k + m)$ such that $v_j \in F_b$, $w_k \in F_c$ and

(i) $(v_j + l, w_k + l + m)$ is an edge in Λ or

(ii) $(u_i \cdot (v_j + l), w_k + l + m)$ is an edge in Λ for some $u_i \in F_a$.

And we let Λ' be the matching between F_a and $F_{b\Theta' c} + l$ that consists of the following three kinds of edges:

(i) $(u_i, v_j + l)$, where $u_i \in F_a$, $v_j \in F_b$ and $(u_i, v_j + l)$ is an edge in Θ;

(ii) $(u_i, (v_j + l) \cdot (w_k + l + m))$, where $u_i \in F_a$, $v_j \in F_b$, $w_k \in F_c$ and $(u_i \cdot (v_j + l), w_k + l + m)$ is an edge in Λ;

(iii) $(u_i, w_k + l + m)$, where $u_i \in F_a$, $w_k \in F_c$ and $(u_i, w_k + l + m)$ is an edge in Λ.

Then it can be checked easily that $(\Theta, \Lambda) \mapsto (\Theta', \Lambda')$ is the desired bijection. The proof is complete. ∎

By the definition of the coproduct and the counit given in (3.4) and (3.5), we can verify the coassociativity and the counitary property without any difficult, so we obtain

Proposition 3.4 *With the coproduct Δ defined by* (3.4) *and counit ε defined by* (3.5), $\mathbb{K}\mathcal{P}$ *is a cocommutative coalgebra.*

By (3.5), it is easy to see that the counit is an algebra map. So it suffices to show that the coproduct map is also an algebra map. To this end, we need two lemmas. We begin with mentioning a simple observation and introducing some notation.

Firstly, a simple observation shows that, for any word a, the relative order of integers in a is invariant under the operator Park. So there is a one-to-one correspondence $\iota : F_{\mathrm{Park}(a)} \to F_a$ such that the relative order of integers in each word of $\iota(u)$ is the same as that of u. In fact, if $\bar{a}$ is the word appearing in the definition of Park(a), then there is a similar bijection between $F_{\bar{a}}$ and F_a. We will use this fact occasionally in the following.

Suppose that a, b are two words on positive integers. For an integer k and a matching Θ_k between F_a and $F_b + k$, let $a\Theta_k b$ denote the word whose LR-decomposition is obtained from F_a and $F_b + k$ by concatenating u and $w + k$ if $(u, w + k)$ is an edge in Θ. Note that, when a and b are parking functions, we use the simpler notation $a\Theta b$ instead of $a\Theta_{l(a)} b$, where $l(a)$ denotes the length of a.

Now assume that $a \in P_m$, $b \in P_n$, $\Theta \in R(a, b)$. Suppose that a' and b' are subwords of a and b respectively such that

$$F_{a'} \subseteq F_a, \quad F_{b'} \subseteq F_b, \quad l(a') = m' \text{ and } l(b') = n'.$$

By restricting Θ to $F_{a'}$ and $F_{b'} + m$, we get a matching Θ_m between $F_{a'}$ and $F_{b'} + m$, so the notation $a'\Theta_m b'$ makes sense. Let Θ' be the matching between $F_{\mathrm{Park}(a')}$ and $F_{\mathrm{Park}(b')} + m'$ with edge set

$$\{(u, w + m') \mid \iota_1(u) \cdot (\iota_2(w) + m) \in F_{a'\Theta_m b'}\},$$

where $\iota_1 : F_{\mathrm{Park}(a')} \to F_{a'}$ and $\iota_2 : F_{\mathrm{Park}(b')} \to F_{b'}$ denote the natural bijections mentioned above. Note that Θ' only depends on Θ_m.

Lemma 3.5 *We have*

$$\mathrm{Park}(a'\Theta_m b') = \mathrm{Park}(a')\Theta'\mathrm{Park}(b').$$

Proof. Firstly, we show the following two claims.

Claim (1): Suppose that $c \in P_{m'}$ and $b = b_1 b_2 \cdots b_k$ is any word on positive integers. If $m \geq m'$, then for any matching Θ_m between F_c and $F_b + m$, we have

$$\mathrm{Park}(c\,\Theta_m b) = c\,\Theta'\mathrm{Park}(b)$$

where Θ' is the matching between F_c and $F_{\mathrm{Park}(b)} + m'$ with edge set

$$\{(u, w + m') \mid u \cdot (\iota(w) + m) \in F_{c\Theta_m b}\}$$

and ι is the natural bijection between $F_{\mathrm{Park}(b)}$ and F_b. We proceed by double induction on m and $\sum_{j=1}^{k} b_j$. It is quite easy to verify the claim for $m = m'$ and $b' = 11 \cdots 1$.

Since $c \in P_{m'}$, we have $d(c\,\Theta_m b) \geq m' + 1$, where $d(x)$ is the integer appearing in the definition of $\mathrm{Park}(x)$. If $d(c\,\Theta_m b) > m'+1$, then there exists some b_j such that $b_j + m \leq m'+1$. It follows that $m = m'$ and $b_j = 1$. In this case, it is easy to see that $d(c\Theta_m b) = d(b) + m$ and

$$\mathrm{Park}(c\,\Theta_m b) = \begin{cases} \mathrm{Park}(c\,\bar{\Theta}_m \bar{b}), & \text{if } b \notin P_k; \\ c\,\Theta_m b, & \text{if } b \in P_k \end{cases}$$

where $\bar{\Theta}_m$ is the matching between F_c and $F_{\bar{b}} + m$ with edge set

$$\{(u, w + m) \mid u \cdot (\bar{\iota}(w) + m) \in F_{c\Theta_m b}\}$$

and $\bar{\iota}$ is the natural bijection between $F_{\bar{b}}$ and F_b. If $b \in P_k$, we are done. Otherwise, we can complete the proof by induction since $\sum_{j=1}^{k} \bar{b}_j < \sum_{j=1}^{k} b_j$ and $\bar{\Theta}' = \Theta'$, where $\bar{\Theta}'$ is the matching between F_c and $F_{\mathrm{Park}(\bar{b})} + m'$ obtained from $\bar{\Theta}_m$ in the same way as that Θ' is obtained from Θ_m.

If $d(c\Theta_m b) = m' + 1$, then $m' < m$ or $m = m'$ and $b_i > 1$ for all $1 \leq i \leq k$. If $m' < m$, by the definition of parkization, we have

$$\mathrm{Park}(c\,\Theta_m b) = \mathrm{Park}(c\,\bar{\Theta}_{m-1} b),$$

where $\bar{\Theta}_{m-1}$ denotes the matching between F_c and $F_b + (m-1)$ with edge set

$$\{(u, w + (m-1)) \mid u \cdot (w + m) \in F_{c\,\Theta_m b}\}.$$

Therefore, we can complete the proof by induction since $m' \leq m - 1$ and $\bar{\Theta}' = \Theta'$. If $m = m'$ and $b_i > 1$ for all $1 \leq i \leq k$, then by the definition of parkization we have

$$\mathrm{Park}(c\,\Theta_m b) = \mathrm{Park}(c\,\bar{\Theta}_m (b-1)),$$

where $b - 1$ denotes the word obtained from b by subtracting 1 from each element and $\bar{\Theta}_m$ denotes the matching between F_c and $F_{b-1} + m$ with edge set

$$\{(u, w + m) \mid u \cdot ((w + 1) + m) \in F_{c\Theta_m b}\}.$$

Again, we have $\bar{\Theta}' = \Theta'$. Since $\sum_{j=1}^{k}(b_j - 1) < \sum_{j=1}^{k} b_j$, by induction, we are done.

Claim (2): Suppose that $c \in P_m$ and $c' = c_{i_1}c_{i_2}\cdots c_{i_{m'}}$ is any subword of c. Let $j(c')$ be the number of steps involved in the algorithm of computing $\mathrm{Park}(c')$, namely,

$$j(c') = \begin{cases} 0, & \text{if } c' \in P_{m'}; \\ j(\bar{c'}) + 1, & \text{otherwise,} \end{cases}$$

where $\bar{c'}$ denotes the word appearing in the definition of $\mathrm{Park}(c')$. Then we have $j(c') \le m - m'$.

The claim is trivially true if $c' \in P_{m'}$. Assume that $c' \notin P_{m'}$. Then $m' < m$ and $j(c') = j(\bar{c'}) + 1$. Let c^* be the word obtained from c by replacing each c_{i_k} by $\bar{c'}_k$ and then deleting a maximal element in the complementary subword $c_{j_1}c_{j_2}\cdots c_{j_{m-m'}}$, where $\{j_1 < j_2 < \cdots < j_{m-m'}\} = [m] \setminus \{i_1, i_2, \ldots, i_{m'}\}$. Clearly, $c^* \in P_{m-1}$ and $\bar{c'}$ is a subword of c^*. By an inductive argument on m, we get that

$$j(c') = j(\bar{c}') + 1 \le m - 1 - m' + 1 = m - m'.$$

Now we are ready to prove the lemma. If $a' \in P_{m'}$, by Claim (1), we are done. Assume that $a' \notin P_{m'}$, then $m' < m$ and $d(a'\Theta_m b') \le m'$. By the definition of parkization, we have

$$\mathrm{Park}(a'\Theta_m b') = \mathrm{Park}(\mathrm{Park}(a')\,\bar{\Theta}_{m-j(a')}\,b')$$

where $j(a')$ denotes the number of steps involved in the algorithm of computing $\mathrm{Park}(a')$ and $\bar{\Theta}_{m-j(a')}$ denotes the matching between $F_{\mathrm{Park}(a')}$ and $F_{b'} + (m - j(a'))$ with edge set

$$\{(u, v + (m - j(a'))) \mid \iota_1(u) \cdot (v + m) \in F_{a'\Theta_m b'}\}, \qquad (\iota_1 : F_{\mathrm{Park}(a')} \to F_{a'}).$$

Clearly, by the construction of Θ' , we have $\bar{\Theta}' = \Theta'$. By Claim (2), we have $m' \le m - j(a')$. Then the proof follows from Claim (1). ∎

Lemma 3.6 *Suppose that $a \in P_m$ and $b \in P_n$. Let*

$$X(a, b) = \{(\Theta, c', c'') \mid \Theta \in R(a, b), F_{c'} \cup F_{c''} = F_{a\Theta b}\}$$

and

$$Y(a, b) = \{(a', a'', b', b'', \Theta', \Theta'') \mid F_{a'} \cup F_{a''} = F_a, F_{b'} \cup F_{b''} = F_b,$$
$$\Theta' \in R(\mathrm{Park}(a'), \mathrm{Park}(b')), \Theta'' \in R(\mathrm{Park}(a''), \mathrm{Park}(b''))\}.$$

Then there exists a bijection

$$X(a, b) \longrightarrow Y(a, b)$$
$$(\Theta, c', c'') \longmapsto (a', a'', b', b'', \Theta', \Theta'')$$

such that

$$\mathrm{Park}(c') = \mathrm{Park}(a')\Theta'\mathrm{Park}(b') \tag{3.7}$$

and

$$\mathrm{Park}(c'') = \mathrm{Park}(a'')\Theta''\mathrm{Park}(b''). \tag{3.8}$$

Proof. Suppose that $F_a = (u_1, u_2 \ldots, u_r)$, $F_b = (w_1, w_2, \ldots, w_s)$, and $(\Theta, c', c'') \in X(a, b)$. Let a', a'', b' and b'' be the words such that

$$F_{a'} = \{u \in F_a \mid u \in F_{c'} \text{ or } u \cdot (w+m) \in F_{c'} \text{ for some } w \in F_b\}$$

$$F_{a''} = \{u \in F_a \mid u \in F_{c''} \text{ or } u \cdot (w+m) \in F_{c''} \text{ for some } w \in F_b\}$$

$$F_{b'} = \{w \in F_b \mid w+m \in F_{c'} \text{ or } u \cdot (w+m) \in F_{c'} \text{ for some } u \in F_a\}$$

$$F_{b''} = \{w \in F_b \mid w+m \in F_{c''} \text{ or } u \cdot (w+m) \in F_{c''} \text{ for some } u \in F_a\}.$$

Then clearly we have $F_{a'} \uplus F_{a''} = F_a$ and $F_{b'} \uplus F_{b''} = F_b$. Assume that $\mathrm{Park}(a') \in P_{m'}$, then $\mathrm{Park}(\mathrm{a}'') \in P_{m-m'}$. Now let Θ' be the matching between $F_{\mathrm{Park}(a')}$ and $F_{\mathrm{Park}(b')} + m'$ with edge set:

$$\{(u', w' + m') \mid \iota_1(u') \cdot (\iota_2(w') + m) \in F_{c'}\},$$

and let Θ'' be the matching between $F_{\mathrm{Park}(a'')}$ and $F_{\mathrm{Park}(b'')} + m - m'$ with edge set:

$$\{(u'', w'' + m - m') \mid \iota_3(u'') \cdot (\iota_4(w'') + m) \in F_{c''}\},$$

where

$$\iota_1 \colon F_{\mathrm{Park}(a')} \to F_{a'}$$

$$\iota_2 \colon F_{\mathrm{Park}(b')} \to F_{b'}$$

$$\iota_3 \colon F_{\mathrm{Park}(a'')} \to F_{a''}$$

$$\iota_4 \colon F_{\mathrm{Park}(b'')} \to F_{b''}$$

are the natural one-to-one correspondences as mentioned before. It is not difficult to verify that $(\Theta, c', c'') \mapsto (a', a'', b', b'', \Theta', \Theta'')$ is a bijection. Moreover, according to our construction, we have

$$c' = a' \Theta_m b', \quad \text{and} \quad c'' = a'' \Theta_m b''.$$

Then equations (3.7) and (3.8) follow immediately from Lemma 3.5. ∎

Proposition 3.7 *For any $a \in P_m$ and $b \in P_n$, we have $\Delta(M_a \star M_b) = \Delta(M_a) \star \Delta(M_b)$.*

Proof. By the definitions of product and coproduct given by (3.3) and (3.4), we have

$$\begin{aligned}
\Delta(M_a \star M_b) = \Delta(\sum_{\Theta \in R(a,b)} M_{a\Theta b}) &= \sum_{\Theta \in R(a,b)} \Delta(M_{a\Theta b}) \\
&= \sum_{\Theta \in R(a,b)} \sum_{F_{c'} \uplus F_{c''} = F_{a\Theta b}} M_{\mathrm{Park}(c')} \otimes M_{\mathrm{Park}(c'')} \\
&= \sum_{(\Theta, c', c'') \in X(a,b)} M_{\mathrm{Park}(c')} \otimes M_{\mathrm{Park}(c'')}
\end{aligned}$$

and

$$\Delta(M_a) \star \Delta(M_b) = (\sum_{F_{a'} \uplus F_{a''} = F_a} M_{\mathrm{Park}(a')} \otimes M_{\mathrm{Park}(a'')}) \star (\sum_{F_{b'} \uplus F_{b''} = F_b} M_{\mathrm{Park}(b')} \otimes M_{\mathrm{Park}(b'')})$$

$$= \sum_{F_{a'} \uplus F_{a''} = F_a} \sum_{F_{b'} \uplus F_{b''} = F_b} M_{\mathrm{Park}(a')} \star M_{\mathrm{Park}(b')} \otimes M_{\mathrm{Park}(a'')} \star M_{\mathrm{Park}(b'')}$$

$$= \sum_{(a',a'',b',b'',\Theta',\Theta'') \in Y(a,b)} M_{\mathrm{Park}(a')\Theta'\mathrm{Park}(b')} \otimes M_{\mathrm{Park}(a'')\Theta''\mathrm{Park}(b'')}.$$

By Lemma 3.6, we deduce that $\Delta(M_a \star M_b) = \Delta(M_a) \star \Delta(M_b)$. ∎

Now Theorem 3.2 follows from Proposition 3.3, Proposition 3.4 and Proposition 3.7.

4 Freeness of PFSym and the Q-basis

The algebra NCSym was originally studied by Wolf in [18], where she aimed to show that this algebra is freely generated. A combinatorial description of the generating set of Wolf has been found by Bergeron, Reutenauer, Rosas, and Zabrocki [5] in terms of monomial basis. The freeness of NCSym can also be proved by using another two bases, the power sum basis introduced by Rosas and Sagan [16] and the q-basis introduced by Bergeron and Zabrocki [6].

In this section, we shall show the freeness of the algebra PFSym. To this end, we introduce the notions of atomic parking function and unsplitable parking function. We prove that PFSym is freely generated by those monomial basis elements indexed by unsplitable parking functions. Using a partial order on parking functions, we introduce a new basis $\{Q_a\}$ which is related to the monomial basis $\{M_a\}$ via Möbius inversion. In terms of this basis, we find another free generating set of PFSym indexed by atomic parking functions. Based on this fact, we deduce that PFSym is isomorphic to the graded dual PQSym* of the Hopf algebra PQSym in [10].

Definition 4.1 *Suppose that $a = a_1a_2\cdots a_m \in P_m$ and $b = b_1b_2\cdots b_n \in P_n$. Let $F_a = (u_1, u_2, \ldots, u_r)$, $F_b = (w_1, w_2, \ldots, w_s)$ and let*

$$F_a \circ F_b = \begin{cases} (u_1 \cdot (w_1 + m), \ldots, u_r \cdot (w_r + m), w_{r+1} + m, \ldots, w_s + m), & \text{if } r \le s; \\ (u_1 \cdot (w_1 + m), \ldots, u_s \cdot (w_s + m), u_{s+1}, \ldots, u_r), & \text{if } r > s. \end{cases} \tag{4.9}$$

By Remark 3.1, we have $F_a \circ F_b \in \mathcal{F}_{m+n}$. So there is a unique parking function, denoted by $a \circ b$, such that $F_{a\circ b} = F_a \circ F_b$. We call $a \circ b$ the split product of a and b.

For example, let $a = 2213 \in P_4$ and $b = 32214 \in P_5$. Then we have $F_a = (13, 22)$, $F_b + 4 = (58, 66, 7)$, and $F_a \circ F_b = (1358, 2266, 7)$. So we have $a \circ b = 722661358$. It can be observed easily that the binary operation $\circ$ is associative and so we can define the split product of $k \ge 2$ parking functions:

$$a^{(1)} \circ a^{(2)} \circ \cdots \circ a^{(k)}$$

in the obvious way.

Definition 4.2 *A parking function a is said to be unsplitable if it can not be written as the split product of two nonempty parking functions.*

Denote by $\mathcal{U}P_n$ the set of unsplitable parking functions of length n. For example, we have

$$\mathcal{U}P_3 := \{111, 112, 121, 131, 132, 211, 212, 311, 221, 231, 321\}.$$

Set $\mathcal{U}P = \cup_{n\geq 1}\mathcal{U}P_n$.

Theorem 4.3 *The algebra* PFSym *is freely generated by the set* $\{M_a \mid a \in \mathcal{U}P\}$.

Proof. Define a total order $\preceq^*$ on the set of words on positive integers as follows. Suppose that $a = a_1a_2\cdots a_i$ and $b = b_1b_2\cdots b_j$, then $a \prec^* b$ if and only if

(1) b is a proper initial subword of a, that is, $i > j$ and $a_l = b_l$ for any $1 \leq l \leq j$ or

(2) a is not a proper initial subword of b and $a <_{\text{lex}} b$, where $<_{\text{lex}}$ denotes the lexicographical order.

For $a, b \in P_n$, define $a \preceq^*_{\text{lex}} b$ if $F_a \preceq^*_{\text{lex}} F_b$, where $\preceq^*_{\text{lex}}$ denotes the lexicographic order induced by $\preceq^*$. That is $(u_1, u_2, \ldots, u_r) \prec^*_{\text{lex}} (w_1, w_2, \ldots, w_s)$ if there exists $i \leq \min(r, s)$ such that $u_j = w_j$ for any $1 \leq j < i$ and $u_i \prec^* w_i$. For example, we have

$$\begin{aligned}
&(111) \prec^*_{\text{lex}} (112) \prec^*_{\text{lex}} (113) \prec^*_{\text{lex}} (11,2) \prec^*_{\text{lex}} (11,3) \prec^*_{\text{lex}} (121) \prec^*_{\text{lex}} (122) \prec^*_{\text{lex}} (123) \prec^*_{\text{lex}}\\
&(12,2) \prec^*_{\text{lex}} (12,3) \prec^*_{\text{lex}} (131) \prec^*_{\text{lex}} (132) \prec^*_{\text{lex}} (13,2) \prec^*_{\text{lex}} (1,22) \prec^*_{\text{lex}} (1,23) \prec^*_{\text{lex}} (1,2,3),
\end{aligned}$$

and so in P_3, we have

$$\begin{aligned}
&111 \prec^*_{\text{lex}} 112 \prec^*_{\text{lex}} 113 \prec^*_{\text{lex}} 211 \prec^*_{\text{lex}} 311 \prec^*_{\text{lex}} 121 \prec^*_{\text{lex}} 122 \prec^*_{\text{lex}} 123 \prec^*_{\text{lex}}\\
&212 \prec^*_{\text{lex}} 312 \prec^*_{\text{lex}} 131 \prec^*_{\text{lex}} 132 \prec^*_{\text{lex}} 213 \prec^*_{\text{lex}} 221 \prec^*_{\text{lex}} 231 \prec^*_{\text{lex}} 321.
\end{aligned}$$

For any parking function a, we can uniquely decompose a into split product of unsplitable parking functions:

$$a = a^{(1)} \circ a^{(2)} \circ \cdots \circ a^{(k)}.$$

Now define $R_a = M_{a^{(1)}} \star M_{a^{(2)}} \star \cdots \star M_{a^{(k)}}$. We claim that

$$R_a = M_a + \sum_{a \prec^*_{\text{lex}} b} c_b M_b. \tag{4.10}$$

The claim can be proved by induction on k. The case when $k = 1$ is trivially true since $R_a = M_a$. Assume that $k > 1$ and the claim is true for any word a' that can be decomposed into split product of $k-1$ unsplitable parking functions. Suppose that $a = a^{(1)} \circ a^{(2)} \circ \cdots \circ a^{(k)} \in P_n$ and $a' = a^{(1)} \circ a^{(2)} \circ \cdots \circ a^{(k-1)} \in P_m$. Then $a = a' \circ a^{(k)}$ and

$$\begin{aligned}
R_a &= R_{a'} \star M_{a^{(k)}}\\
&= (M_{a'} + \sum_{a' \prec^*_{\text{lex}} b'} c_{b'} M_{b'}) \star M_{a^{(k)}}
\end{aligned}$$

$$= \sum_{\Theta \in R(a', a^{(k)})} M_{a'\Theta a^{(k)}} + \sum_{a' \prec^*_{\text{lex}} b'} \sum_{\Theta \in R(b', a^{(k)})} c_{b'} M_{b'\Theta a^{(k)}}.$$

To complete the proof, it suffices to show the following facts:

(1) $a' \circ a^{(k)} = a' \Lambda a^{(k)}$ for a unique $\Lambda \in R(a', a^{(k)})$;

(2) For any $\Theta \in R(a', a^{(k)})$, if $\Theta \neq \Lambda$, then $a' \circ a^{(k)} \prec^*_{\text{lex}} a'\Theta a^{(k)}$;

(3) If $a' \prec^*_{\text{lex}} b'$ in P_m, then $a' \circ a^{(k)} \prec^*_{\text{lex}} b' \circ a^{(k)}$ in P_n.

Assume that $F_{a'} = (u_1, u_2, \ldots, u_r), F_{a^{(k)}} = (v_1, v_2, \ldots, v_s)$. Obviously, the matching Λ with edge set

$$\{(u_i, v_i + m) \mid 1 \leq i \leq \min(r, s)\}$$

is the desired matching in fact (1). For any $\Theta \in R(a', a^{(k)})$, let $F_{a'\Theta a^{(k)}} = (w_1, w_2, \ldots, w_t)$. A simple observation shows that if $i \leq r$, then $w_i = u_i$ or $w_i = u_i \cdot (v_j + m)$ for some $1 \leq j \leq s$. If $\Theta \neq \Lambda$, then there exists some $h \leq \min(r, s)$ such that $w_h \neq u_h \cdot (v_h + m)$. We choose h to be the smallest integer satisfying this property. Then $w_h = u_h$ or $w_h = u_h \cdot (v_j + m)$ for some $j > h$. In either case, we have $u_h \cdot (v_h + m) \prec^* w_h$. By the definition of $\prec^*_{\text{lex}}$, we deduce that $a' \circ a^{(k)} \prec^*_{\text{lex}} a'\Theta a^{(k)}$. This completes the proof of fact (2). We continue to prove fact (3). If $a' \prec^*_{\text{lex}} b'$ in P_m, then we can assume that

$$F_{b'} = (u_1, u_2, \ldots, u_{h-1}, u_h^{'}, \ldots, u_l^{'}),$$

where $h \leq r$ and $u_h \prec^* u_h^{'}$. This implies that if $F_{a' \circ a^{(k)}} = (w_1, w_2, \ldots, w_{h-1}, w_h, \ldots, w_t)$, then $F_{b' \circ a^{(k)}}$ takes the form

$$(w_1, w_2, \ldots, w_{h-1}, w_h', \ldots, w_j'),$$

where

$$w_h' = \begin{cases} u_h', & \text{if } w_h = u_h; \\ u_h' \cdot (v_h + m), & \text{if } w_h = u_h \cdot (v_h + m). \end{cases}$$

In either case, it is easy to show that $w_h \prec^* w_h'$. Hence we have $a' \circ a^{(k)} \prec^*_{\text{lex}} b' \circ a^{(k)}$. This proves fact (3).

Therefore the claim (4.10) is true. By triangularity, we deduce that the set $\cup_{n \geq 0}\{R_a \mid a \in P_n\}$ is also a basis for $\mathbb{K}\mathcal{P}$. So by the definition of R_a, we conclude that

$$\{M_a \mid a \text{ is unsplitable}\}$$

freely generates the algebra PFSym. ∎

From the proof of Theorem 4.3, we deduce that $\cup_{n \geq 0}\{R_a \mid a \in P_n\}$ is a multiplicative basis of PFSym since by the definition of R-basis, we have $R_a \star R_b = R_{a \circ b}$. We now introduce another multiplicative basis. To begin with, we introduce the slash product of parking functions and the notion of atomic parking function.

Definition 4.4 *Suppose that $a = a_1 a_2 \cdots a_m \in P_m$ and $b = b_1 b_2 \cdots b_n \in P_n$. Let $a|b = (b_1 + m)(b_2 + m) \cdots (b_n + m) a_1 a_2 \cdots a_m$. We call $a|b$ the slash product of a and b.*

Let $F_a = (u_1, u_2, \ldots, u_r)$, $F_b = (w_1, w_2, \ldots, w_s)$ and let

$$F_a|F_b = (u_1, u_2, \ldots, u_r, w_1 + m, w_2 + m, \ldots, w_s + m).$$

Clearly, we have $F_{a|b} = F_a|F_b$. So the slash product of parking functions can be seen as a natural generalization of the slash product of set partitions [7]. Note that the slash product $|$ is also associative.

Definition 4.5 *A parking function a is said to be atomic if there are no nonempty parking functions b and c such that $a = b|c$.*

Note that the notion of atomic parking function is related to the notion of prime parking function introduced by Gessel in 1977, see[14] for details. Let $\mathcal{A}P_n$ denote the set of atomic parking functions of length n. For example, we have

$$\mathcal{A}P_3 = \{111, 211, 121, 131, 112, 212, 122, 132, 113, 213, 123\}.$$

Set $\mathcal{A}P = \cup_{n\geq 1}\mathcal{A}P_n$

Now we define a partial order $\leq_*$ on P_n and then introduce a new basis $\{Q_a\}$ via Möbius inversion. Suppose that $a, b \in P_n$ and $F_a = (w_1, w_2, \ldots, w_r)$. We say b covers a if there exist $1 \leq i < j \leq r$ such that every element in w_i is less than or equal to every element in w_j and

$$F_b = (w_1, \ldots, w_{i-1}, w_i \cdot w_j, w_{i+1}, \ldots, w_{j-1}, w_{j+1}, \ldots, w_r).$$

Let $\leq_*$ denote the partial order on P_n generated by these covering relations. For example, the following figure shows the Hasse diagram of $(P_3, \leq_*)$.

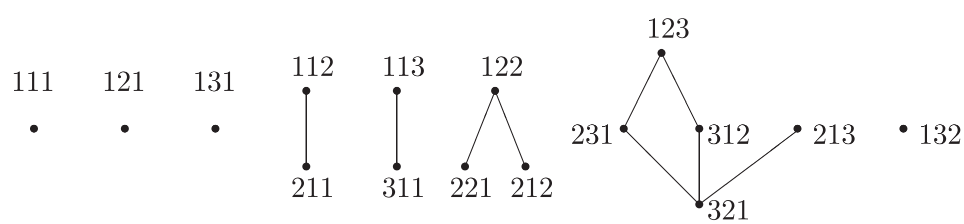


Fig.1. the poset $(P_3, \leq_*)$

For any parking function $a \in P_n$, set

$$Q_a = \sum_{a \leq_* b} M_b. \tag{4.11}$$

By Möbius inversion, $\cup_{n\geq 0}\{Q_a \mid a \in P_n\}$ is also a basis of $\mathbb{K}\mathcal{P}$. Moreover, the following theorem shows that it is a multiplicative basis.

Theorem 4.6 *Suppose that $a \in P_m$ and $b \in P_n$. Then*

$$Q_a \star Q_b = Q_{a|b}, \tag{4.12}$$

and

$$\Delta(Q_a) = \sum_{F_{a'} \uplus F_{a''} = F_a} Q_{\mathrm{Park}(a')} \otimes Q_{\mathrm{Park}(a'')}. \tag{4.13}$$

In particular, the set $\{Q_a \mid a \in \mathcal{A}P\}$ freely generates the algebra PFSym.

Proof. By the definitions of Q_a and $\star$, we have

$$\begin{aligned} Q_a \star Q_b &= (\sum_{a \le_* \tilde{a}} M_{\tilde{a}}) \star (\sum_{b \le_* \tilde{b}} M_{\tilde{b}}) \\ &= \sum_{a \le_* \tilde{a},\ b \le_* \tilde{b}} M_{\tilde{a}} \star M_{\tilde{b}} \\ &= \sum_{a \le_* \tilde{a},\ b \le_* \tilde{b}} \ \sum_{\Theta \in R(\tilde{a}, \tilde{b})} M_{\tilde{a}\Theta\tilde{b}}, \end{aligned} \tag{4.14}$$

and

$$Q_{a|b} = \sum_{a|b \le_* c} M_c. \tag{4.15}$$

According to (4.14) and (4.15), for any $c' \in P_{m+n}$, the coefficient of $M_{c'}$ in $Q_a \star Q_b$ or in $Q_{a|b}$ is 0 or 1. So it suffices to show that if

$$A(a,b) = \{\tilde{a}\Theta\tilde{b} :\ \tilde{a} \in P_m, \tilde{b} \in P_n, a \le_* \tilde{a}, b \le_* \tilde{b}, \Theta \in R(\tilde{a}, \tilde{b})\}$$

and

$$B(a,b) = \{c :\ c \in P_{m+n}, a|b \le_* c\},$$

then $A(a,b) = B(a,b)$. Let $F_a = (u_1, u_2, \ldots, u_r)$ and $F_b = (w_1, w_2, \ldots, w_s)$. Suppose that $c = \tilde{a}\Theta\tilde{b} \in A(a,b)$. Since $a \le_* \tilde{a}$ and $b \le_* \tilde{b}$, we deduce that each word v in $F_{\tilde{a}}$ takes the form

$$v = u_{i_1} \cdot u_{i_2} \cdots u_{i_k},\ 1 \le i_1 < i_2 < \cdots < i_k \le r$$

and each word v' in $F_{\tilde{b}}$ takes the form

$$v' = w_{j_1} \cdot w_{j_2} \cdots w_{j_l},\ 1 \le j_1 < j_2 < \cdots < j_l \le s$$

where each integer in u_{i_t} is less than or equal to each integer in $u_{i_{t+1}}$, and each integer in w_{j_t} is less than or equal to each integer in $w_{j_{t+1}}$. So each word v^* in F_c takes one of the following three forms:

(1) $v^* = u_{i_1} \cdot u_{i_2} \cdots u_{i_k}$

(2) $v^* = (w_{j_1} + m) \cdot (w_{j_2} + m) \cdots (w_{j_l} + m)$

(3) $v^* = u_{i_1} \cdot u_{i_2} \cdots u_{i_k} \cdot (w_{j_1} + m) \cdot (w_{j_2} + m) \cdots (w_{j_l} + m)$.

By the definition of $\leq_*$, we have $a|b \leq_* c$ and so $c \in B(a,b)$. Conversely, if $c \in B(a,b)$, then each word v^* in F_c takes one of the above three forms. Let $\tilde{a}, \tilde{b}$ be the parking functions determined by

$$F_{\tilde{a}} = \{v \mid v \in F_c \text{ and } \max(v) \leq m\} \cup \{v \mid v \cdot (v' + m) \in F_c \text{ for some } v' \text{ and } \max(v) \leq m\}$$

$$F_{\tilde{b}} = \{v' \mid v' + m \in F_c\} \cup \{v' \mid v \cdot (v' + m) \in F_c \text{ for some } v \text{ such that } \max(v) \leq m\}.$$

Clearly, we have $a \leq_* \tilde{a}$, $b \leq_* \tilde{b}$ and $c = \tilde{a}\Theta\tilde{b}$, where Θ is the matching between $F_{\tilde{a}}$ and $F_{\tilde{b}} + m$ with edge set

$$\{(v, v' + m) \mid v \cdot (v' + m) \in F_c\}.$$

So we have $c \in A(a,b)$. Hence, $A(a,b) = B(a,b)$ and the equation (4.12) holds.

By the definition of Q_a, we have

$$\begin{aligned}\Delta(Q_a) = \Delta(\sum_{a \leq_* b} M_b) &= \sum_{a \leq_* b} \Delta(M_b)\\ &= \sum_{a \leq_* b} \sum_{F_{b'} \uplus F_{b''} = F_b} M_{\mathrm{Park}(b')} \otimes M_{\mathrm{Park}(b'')}\end{aligned}$$

and

$$\sum_{F_{a'} \uplus F_{a''} = F_a} Q_{\mathrm{Park}(a')} \otimes Q_{\mathrm{Park}(a'')} = \sum_{F_{a'} \uplus F_{a''} = F_a} \sum_{\substack{\mathrm{Park}(a') \leq_* c' \\ \mathrm{Park}(a'') \leq_* c''}} M_{c'} \otimes M_{c''}.$$

Let

$$X(a) := \{(b, b', b'') \mid a \leq_* b, F_{b'} \uplus F_{b''} = F_b\}$$

and

$$Y(a) := \{(a', a'', c', c'') \mid F_{a'} \uplus F_{a''} = F_a, \mathrm{Park}(a') \leq_* c', \mathrm{Park}(a'') \leq_* c''\}.$$

To show (4.13), it suffices to find a bijection $\phi : (b, b', b'') \to (a', a'', c', c'')$ between $X(a)$ and $Y(a)$ such that

$$\mathrm{Park}(b') = c', \quad \mathrm{Park}(b'') = c''.$$

Let $F_a = (u_1, u_2, \ldots, u_r)$ and $(b, b', b'') \in X(a)$. Since $a \leq_* b$, we deduce that every word $v \in F_b$ has the form

$$v = u_{i_1} \cdot u_{i_2} \cdots u_{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq r$$

where each integer in u_{i_t} is less than or equal to each integer in $u_{i_{t+1}}$. Let

$$U(v) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$$

and let a', a'' be the subwords of a such that

$$F_{a'} = \bigcup_{v \in F_{b'}} U(v), \quad F_{a''} = \bigcup_{v \in F_{b''}} U(v).$$

Clearly, $F_{a'} \cap F_{a''} = \emptyset$ and $F_{a'} \cup F_{a''} = F_a$. Recall that there are natural bijections $\iota_1 : F_{\mathrm{Park}(a')} \to F_{a'}$ and $\iota_2 : F_{\mathrm{Park}(a'')} \to F_{a''}$. For $v = u_{i_1} \cdot u_{i_2} \cdots u_{i_k} \in F_b$, we let

$$\iota(v) = \begin{cases} \iota_1^{-1}(u_{i_1}) \cdot \iota_1^{-1}(u_{i_2}) \cdots \iota_1^{-1}(u_{i_k}), & \text{if } v \in F_{b'}; \\ \iota_2^{-1}(u_{i_1}) \cdot \iota_2^{-1}(u_{i_2}) \cdots \iota_2^{-1}(u_{i_k}), & \text{if } v \in F_{b''}, \end{cases}$$

and let c', c'' be the parking functions of length $l(a')$ and $l(a'')$ respectively such that

$$F_{c'} = \{\iota(v) \mid v \in F_{b'}\}, \quad F_{c''} = \{\iota(v) \mid v \in F_{b''}\}.$$

Keeping in mind that the operator Park doesn't change relative order of integers in a word, we can easily verify that $(a', a'', c', c'') \in Y(a)$ and $\phi : (b, b', b'') \to (a', a'', c', c'')$ is the desired bijection. This completes the proof of equation (4.13).

Since each parking function $a \in P$ can be uniquely decomposed into slash product of atomic parking functions, it follows from (4.12) that the algebra PFSym is freely generated by $\{Q_a \mid a \in \mathcal{A}P\}$. ∎

Let $U(L(X))$ denote the universal enveloping algebra of the free Lie algebra $L(X)$ on the set X. By Theorem 4.3, Theorem 4.6 and the Milnor-Moore theorem (see [13, p. 244] or [17, p. 274]), we find that

$$\mathrm{PFSym} \cong U(L(\mathcal{A}P)) \cong U(L(\mathcal{U}P)).$$

To establish a more explicit isomorphism, one can use the method given by Lauve and Mastnak [11] to find the algebraically independent generators of the Lie algebra of primitive elements of PFSym.

Recall that a parking function $a \in P_n$ is connected if there are no nonempty parking functions b and c such that

$$a = b \cdot (c + l(b)),$$

where $l(b)$ denotes the length of b. Let $\mathcal{C}P$ be the set of all connected parking functions and let PQSym* be the graded dual of the commutative Hopf algebra PQSym introduced by Hivert, Novelli and Thibon [10]. By a similar argument as [10, Theorem 2.5], we have PQSym$^* \cong U(L(\mathcal{C}P))$. Therefore, by the obvious fact that $a = a_1 a_2 \cdots a_n \in \mathcal{C}P$ if and only if $\hat{a} = a_n a_{n-1} \cdots a_1 \in \mathcal{A}P$, we conclude that PFSym $\cong$ PQSym*.

To conclude this section, we remark that the Hopf algebra PFSym is also cofree. In fact, since each parking function a can be uniquely decomposed into slash product of atomic parking functions:

$$a = a^{(1)} | a^{(2)} | \cdots | a^{(r)},$$

we can identify parking functions with words on $\mathcal{A}P$. By rephrasing the proof given in [6, Section 4], one can show that PFSym* is isomorphic to a shuffle algebra on a vector space with a basis indexed by atomic parking functions. Therefore the freeness of PFSym*, and so the cofreeness of PFSym, follows from [15, Theorem 6.1].

5 Hopf subalgebras of PFSym

In this section, we discuss Hopf subalgebras of PFSym. As will be seen, the Hopf algebra NCSym can be embedded as a Hopf subalgebra into PFSym in a natural way. Moreover, when

restricting to permutations and non-increasing parking functions, we find two Hopf subalgebras isomorphic to the Grossman-Larson Hopf algebras of ordered trees and heap-ordered trees respectively.

For each $n \geq 0$, let

$$N_n = \{a \in P_n |\text{ each word in } F_a \text{ is nondecreasing }\}$$

$$D_n = \{a \in P_n |\ F_a = (w_1, w_2, \ldots, w_k) \text{ and } w_i \cap w_j = \emptyset\}$$

$$\mathfrak{S}_n = \{a \in P_n |\ a \text{ is a permutation of } [n]\}$$

$$C_n = \{a \in P_n |\ a = a_1 a_2 \cdots a_n \text{ with } a_1 \geq a_2 \geq \cdots \geq a_n\}$$

where we use $w_i \cap w_j$ to denote the intersection of the underlying sets of w_i and w_j. Then we have

$$\mathfrak{S}_n \subseteq D_n \quad \text{and} \quad C_n \subseteq N_n \cap D_n.$$

Moreover, it is easy to check that each $a \in C_n$ is a minimal element in the poset $(P_n, \leq_*)$ and

$$\{b \in P_n |\ b \geq_* a \text{ for some } a \in C_n\} = N_n \cap D_n. \tag{5.16}$$

Now set

$$\mathcal{A}C_n = \mathcal{A}P_n \cap C_n, \quad \mathcal{A}C = \cup_{n\geq 1} \mathcal{A}C_n,$$
$$\mathcal{A}N_n = \mathcal{A}P_n \cap N_n, \quad \mathcal{U}N_n = \mathcal{U}P_n \cap N_n, \quad \mathcal{A}N = \cup_{n\geq 1} \mathcal{A}N_n, \quad \mathcal{U}N = \cup_{n\geq 1} \mathcal{U}N_n,$$
$$\mathcal{A}D_n = \mathcal{A}P_n \cap D_n, \quad \mathcal{U}D_n = \mathcal{U}P_n \cap D_n, \quad \mathcal{A}D = \cup_{n\geq 1} \mathcal{A}D_n, \quad \mathcal{U}D = \cup_{n\geq 1} \mathcal{U}D_n,$$
$$\mathcal{A}\mathfrak{S}_n = \mathcal{A}P_n \cap \mathfrak{S}_n, \quad \mathcal{U}\mathfrak{S}_n = \mathcal{U}P_n \cap \mathfrak{S}_n, \quad \mathcal{A}\mathfrak{S} = \cup_{n\geq 1} \mathcal{A}\mathfrak{S}_n, \quad \mathcal{U}\mathfrak{S} = \cup_{n\geq 1} \mathcal{U}\mathfrak{S}_n,$$

and set

$$\mathbb{K}\mathcal{N} = \bigoplus_{n\geq 0} \mathbb{K}\{M_a \mid a \in N_n\}, \qquad \mathbb{K}\mathcal{D} = \bigoplus_{n\geq 0} \mathbb{K}\{M_a \mid a \in D_n\},$$

$$\mathbb{K}\mathfrak{S} = \bigoplus_{n\geq 0} \mathbb{K}\{M_a \mid a \in \mathfrak{S}_n\}, \qquad \mathbb{K}\mathcal{C} = \bigoplus_{n\geq 0} \mathbb{K}\{Q_a \mid a \in C_n\}.$$

Theorem 5.1 *All the subspaces $\mathbb{K}\mathcal{N}$, $\mathbb{K}\mathcal{D}$, $\mathbb{K}\mathfrak{S}$ and $\mathbb{K}\mathcal{C}$ are Hopf subalgebras of* PFSym*. Moreover, we have*

(1) the Hopf algebra $\mathbb{K}\mathcal{N}$ has two free generating sets: $\{M_a \mid a \in \mathcal{U}N\}$ and $\{Q_a \mid a \in \mathcal{A}N\}$;

(2) the Hopf algebra $\mathbb{K}\mathcal{D}$ has two free generating sets: $\{M_a \mid a \in \mathcal{U}D\}$ and $\{Q_a \mid a \in \mathcal{A}D\}$;

(3) the Hopf algebra $\mathbb{K}\mathfrak{S}$ has two free generating sets: $\{M_a \mid a \in \mathcal{U}\mathfrak{S}\}$ and $\{Q_a \mid a \in \mathcal{A}\mathfrak{S}\}$;

(4) the Hopf algebra $\mathbb{K}\mathcal{C}$ is freely generated by $\{Q_a \mid a \in \mathcal{A}C\}$.

Proof. Suppose that $a \in N_m$ (resp. D_m, $\mathfrak{S}_m$) and $b \in N_n$ (resp. D_n, $\mathfrak{S}_n$). It is easy to show that

$$a\Theta b \in N_{m+n} \quad (\text{resp. } D_{m+n}, \mathfrak{S}_{m+n})$$

for any $\Theta \in R(a,b)$. So the product is closed on $\mathbb{K}\mathcal{N}$ (resp. $\mathbb{K}\mathcal{D}$, $\mathbb{K}\mathfrak{S}$) . Since the operator Park preserve the relative order of integers in words, the coproduct is also closed on $\mathbb{K}\mathcal{N}$ (resp. $\mathbb{K}\mathcal{D}$, $\mathbb{K}\mathfrak{S}$). Now we consider the space $\mathbb{K}\mathcal{C}$. Note that C_n consists of parking functions $a \in P_n$ such that each word in F_a is a word on a singleton set. So for any $a \in C_m, b \in C_n$ and subword a' of a such that $F_{a'} \subseteq F_a$, we have

$$a|b \in C_{m+n} \quad \text{and} \quad \mathrm{Park}(a') \in C_k \qquad \text{for some } k \leq m.$$

Hence it follows from (4.12) and (4.13) that

$$\mathbb{K}\mathcal{C} \cdot \mathbb{K}\mathcal{C} \subseteq \mathbb{K}\mathcal{C}$$

and

$$\Delta(\mathbb{K}\mathcal{C}) \subseteq \mathbb{K}\mathcal{C} \otimes \mathbb{K}\mathcal{C}.$$

So $\mathbb{K}\mathcal{N}$, $\mathbb{K}\mathcal{D}$, $\mathbb{K}\mathfrak{S}$ and $\mathbb{K}\mathcal{C}$ are Hopf subalgebras of PFSym.

Note that for each $n \geq 1$, the set N_n is a dual order ideal of $(P_n, \leq_*)$, i.e., if $a \in N_n$,$b \in P_n$ and $b \geq_* a$, then $b \in N_n$. So we have $\cup_{n\geq 0}\{Q_a \mid a \in N_n\}$ is also a basis for $\mathbb{K}\mathcal{N}$. Now the statement (1) follows from the proof of Theorem 4.3 and equation (4.12). Similarly, we can prove the statements (2) and (3). Using equation (4.12) again, we get the proof of statement (4). ∎

It should be noted that C_n is not a dual order ideal of $(P_n, \leq_*)$ in general. In fact, by (5.16), C_n consists of minimal elements of the dual order ideal $N_n \cap D_n$. So $M_a \notin \mathbb{K}\mathcal{C}$ for any $a \in C_n$ unless $a = 11\cdots 1$.

Corollary 5.2 *Let $\tilde{\Pi}_n = N_n \cap \mathfrak{S}_n$. Then*

$$\mathbb{K}\tilde{\Pi} = \bigoplus_{n\geq 0} \mathbb{K}\{M_a \mid a \in \tilde{\Pi}_n\}$$

is a Hopf subalgebra of PFSym *which is isomorphic to the Hopf algebra* NCSym.

Proof. By Theorem 5.1, $\mathbb{K}\tilde{\Pi}$ is a Hopf subalgebra of PFSym. We proceed to construct an explicit isomorphism between NCSym and $\mathbb{K}\tilde{\Pi}$. For any finite set of integers B, let $w(B)$ denote the increasing arrangement of elements in B. Suppose that $\pi = \{B_1, B_2, \cdots, B_k\} \in \Pi_n$. Let $\omega(\pi) = w(B_k) \cdot w(B_{k-1}) \cdots w(B_1)$, where $\cdot$ denotes concatenation of words. Set $\omega(\emptyset) = \epsilon$. It can be checked easily that, for each $n \geq 0$, $\omega : \Pi_n \to \tilde{\Pi}_n$ is a bijection. Now define a linear map $\bar{\omega}$: NCSym $\to \mathbb{K}\tilde{\Pi}$ by setting $\bar{\omega}(M_\pi) = M_{\omega(\pi)}$. It is straightforward to show that $\bar{\omega}$ is a Hopf algebra isomorphism. ∎

Note that the subposet $(\tilde{\Pi}_n, \leq_*)$ of $(P_n, \leq_*)$ is isomorphic to the poset $(\Pi_n, \leq_*)$ introduced by Bergeron and Zabrocki [6]. As a matter of fact, it is easy to see that the bijection $\omega : \Pi_n \to \tilde{\Pi}_n$ given in the proof of Corollary 5.2 is order-preserving. Therefore, the basis $\{Q_a \mid a \in \cup_{n\geq 0}\tilde{\Pi}_n\}$ for $\mathbb{K}\tilde{\Pi}$ corresponds to the basis $\{\mathbf{q}_\pi \mid \pi \in \cup_{n\geq 0}\Pi_n\}$ for NCSym.

By Theorem 5.1 and the Milnor-Moore theorem, we find that

$$\mathbb{K}\mathcal{N} \cong U(L(\mathcal{A}\mathcal{N})) \qquad \mathbb{K}\mathcal{D} \cong U(L(\mathcal{A}\mathcal{D})),$$

$$\mathbb{K}\mathfrak{S} \cong U(L(\mathcal{A}\mathfrak{S})) \qquad \mathbb{K}\mathcal{C} \cong U(L(\mathcal{A}\mathcal{C})),$$

where $L(X)$ denotes the free Lie algebra on the set X and $U(L)$ denotes the universal enveloping algebra of L. The Hopf algebras $\mathbb{K}\mathcal{N}$ and $\mathbb{K}\mathcal{D}$ seem to be new to our knowledge, whereas the Hopf algebras $\mathbb{K}\mathfrak{S}$ and $\mathbb{K}\mathcal{C}$ are well-known combinatorial Hopf algebras studied in different ways from the literatures.

Corollary 5.3 *The Hopf algebra $\mathbb{K}\mathfrak{S}$ is isomorphic to the Grossman-Larson Hopf algebra $\mathcal{H}_{HO}$ of heap-ordered trees.*

Proof. By Theorem 5.1 and the Milnor-Moore theorem, we find that

$$\mathbb{K}\mathfrak{S} \cong U(L(\mathcal{A}\mathfrak{S})).$$

Let $\mathfrak{S}\mathbf{QSym}^*$ denote the graded dual of the Hopf algebra $\mathfrak{S}$QSym in [10]. By [10, Theorem 3.4]), we have

$$\mathfrak{S}\mathbf{QSym}^* \cong U(L(X)),$$

where

$$X = \cup_{n\geq 1}\{\pi \in \mathfrak{S}_n \mid \pi_1\pi_2\cdots\pi_i \notin \mathfrak{S}_i \text{ for any } 1 \leq i \leq n-1\},$$

denotes the set of connected permutations. Since a permutation $a_1a_2\cdots a_n$ is atomic if and only if $a_na_{n-1}\cdots a_1$ is connected, we deduce that

$$\mathbb{K}\mathfrak{S} \cong \mathfrak{S}\mathbf{QSym}^*.$$

Now the conclusion follows from [10, Corollary 3.5], which states that $\mathfrak{S}\mathbf{QSym}^* \cong \mathcal{H}_{HO}$. ∎

We remark that there are another two Hopf algebras $\Phi\mathbf{Sym}$ (see [10, Proposition 4.3]) and $\mathbb{K}\tilde{\mathfrak{S}}$ (see [9], where the authors use the notation $\mathbb{K}\mathfrak{S}$ we use here) built on permutations which are isomorphic to the Hopf algebra $\mathcal{H}_{HO}$. Their coalgebra structures are defined in a similar way to that of $\mathbb{K}\mathfrak{S}$. More specifically, for a permutation a with $F_a = \{w_1, w_2, \ldots, w_k\}$, let $\varphi(a)$ be the permutation $(w_1)(w_2)\cdots(w_k)$ which is expressed as a product of cycles. Then the map φ induces a coalgebra isomorphism $M_a \mapsto \varphi(a)$ from $\mathbb{K}\mathfrak{S}$ to $\Phi\mathbf{Sym}$ or $\mathbb{K}\tilde{\mathfrak{S}}$. However, φ is not an algebra map since the product of $\mathbb{K}\mathfrak{S}$ is different from both the product given by cyclic shuffle associated with matchings on cycles and the heap product by Grossman and Larson.

As for the Hopf algebra $\mathbb{K}\mathcal{C}$, we have

Corollary 5.4 *The Hopf algebra $\mathbb{K}\mathcal{C}$ is isomorphic to the Grossman-Larson Hopf algebra $\mathcal{H}_O$ of ordered trees.*

Proof. Recall that the number $|C_n|$ of non-increasing parking functions of length n is given by the nth Catalan number. Since it can be shown easily that

$$\mathcal{A}C_n = \{u \cdot 1 \mid u \in C_{n-1}\},$$

it follows that the set $\mathcal{A}C_n$ is enumerated by the $(n-1)$th Catalan number. So we deduce that, as a free algebra generated by the Catalan set $\{Q_a \mid a \in \mathcal{A}C\}$, the Hopf algebra $\mathbb{K}\mathcal{C}$ must be isomorphic to the Grossman-Larson Hopf algebra $\mathcal{H}_O$ of ordered trees [8]. ∎

The Hopf algebra $\mathcal{H}_O$ also appears as the graded dual of the quotient of the Loday-Ronco Hopf algebra of planar binary trees by its coradical filtration [2, 3, 12]. Besides, this Hopf algebra is also isomorphic to the Catalan Quasi-symmetric Hopf algebra CQSym, which is a Hopf subalgebra of the non-cocommutative Hopf algebra of parking functions introduced by Novelli and Thibon [14]. So we have obtained a new way to approach these Hopf algebras.

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